Mechanics of solids

UNIT-3 Flexural and Shear Stresses in Beams

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CONTENTS:

- Bending in Beams
- Stresses due to bending
- Simple Bending or Pure bending
- Assumptions of theory of Simple Bending
- Neutral Axis
- Moment of Resistance
- Bending of Flitched Beams
- Shear stresses in beams

FLEXURAL STRESSES: Theory of simple bending – Assumptions – Derivation of bending equation: \( \frac{M}{I} = \frac{f}{y} = \frac{E}{R} \)
Neutral axis – Determination bending stresses – section modulus of rectangular and circular sections (Solid and Hollow), I, T, Angle and Channel sections – Design of simple beam sections.

SHEAR STRESSES: Derivation of formula – Shear stress distribution across various beams sections like rectangular, circular, triangular, I, T angle sections.
Bending in beams

Before deformation

(a)

After deformation

(b)

Horizontal lines become curved

Vertical lines remain straight, yet rotate
Stresses due to bending

INITIAL SHAPE OF PLANKS
WHAT THE LOADS DO????

Bending

Compression

Tension

Shear
PURE BENDING or SIMPLE BENDING

SFD and BMD for OVER HANGING BEAM POINT LOAD with Point Load

Base Line

Base Line

KNm

KNm

Beam subjected to bending moment only with zero S.F.
Simple bending or Pure bending

• When some external force acts on a beam, the shear force and bending moments are set up at all the sections of the beam.

• Due to shear force and bending moment, the beam undergoes deformation. The material of the beam offers resistance to deformation.

• Stresses introduced by bending moment are known as bending stresses.

• These Bending stresses are indirect normal stresses.
Simple bending or Pure bending

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• Stresses introduced by bending moment are known as bending stresses.
• These Bending stresses are indirect normal stresses.
Theory of simple bending

Assumptions

The following are the assumptions of Simple Bending:

1. The material of the beam is isotropic and homogeneous. I.e., of same density and elastic properties throughout.

2. The value of Young’s modulus of elasticity is the same in tension and compression.

3. The transverse sections which were plane before bending, remain plane after bending also.

4. The beam is initially straight and all longitudinal filaments bend into circular arcs with a common centre of curvature.

5. The radius of curvature is large compared with the dimensions of the cross-section.

6. Each layer of the beam is free to expand or contract, independently of the layer, above or below it.
Theory of simple bending

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Theory of simple bending

- Consider a beam subjected to simple bending. Consider an infinitesimal element of length $dx$ which is a part of this beam. Consider two transverse sections AB and CD which are normal to the axis of the beam and parallel to each other.
Theory of simple bending

• Due to the bending action the element ABCD is deformed to A’B’C’D’ (concave curve).
• The layers of the beam are not of the same length before bending and after bending.
• The layer AC is shortened to A’C’. Hence it is subjected to compressive stress.
• The layer BD is elongated to B’D’. Hence it is subjected to tensile stresses.
• Hence the amount of shortening decrease from the top layer towards bottom and the amount of elongation decreases from the bottom layer towards top.
• Therefore there is a layer in between which neither elongates nor shortens. This layer is called neutral layer.
Neutral Axis

• For a beam subjected to a pure bending moment, the stresses generated on the neutral layer is zero.

• Neutral axis is the line of intersection of neutral layer with the transverse section

• Consider the cross section of a beam subjected to pure bending. The stress at a distance $y$ from the neutral axis is given by $\sigma/y = E/R$
Neutral Axis

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Stresses due to bending

Strain in layer EF \( = \frac{y}{R} \)

\[
E = \frac{\text{Stress in the layer EF}}{\text{Strain in the layer EF}}
\]

\[
E = \frac{\sigma}{\left(\frac{y}{R}\right)}
\]

\[
\frac{\sigma}{y} = \frac{E}{R} \quad \Rightarrow \quad \sigma = \frac{E}{R}y
\]
REMEMBER POINT

**Flexure Formula**

\[
\frac{M}{I} = \frac{E}{R} = \frac{\sigma}{y}
\]
Derivation of Flexure Formula or Bending Equation \(\frac{M}{I} = \frac{F}{Y} = \frac{E}{R}\)

\(\text{Where } M = \text{Bending Moment} \quad \text{in}-\text{mm}\)

\(I = \text{Moment of Inertia} \quad \text{mm}^4\)

\(F = \text{Bending Stress} \quad \text{N/mm}^2\)

\(Y = \text{Distance of the layer from the Neutral Axis} \quad \text{mm}\)

\(E = \text{Young's Modulus of the Material} \quad \text{N/mm}^2\)

\(R = \text{Radius of Curvature} \quad \text{mm}\)

Two Steps:

1. Strains
2. Moment of Inertia

Strain = \(\varepsilon = \frac{\text{Elongated length}}{\text{Original length}}\)

\[E = \frac{F}{E} \Rightarrow \varepsilon = \frac{F}{E}\]

\[\varepsilon = \frac{y}{R} \Rightarrow \frac{F}{E} = \frac{y}{R}\]

\[\varepsilon = \frac{y}{R}\]
2. **Step**

- **Load**
  \[ L = f \times dA \]
  \( \text{Load} \)
  \[ = \left( \frac{E \cdot y}{R} \right) \times dA \quad \text{\( \because \) from (3)} \]

- **Load about the element**
  \[ M_{N-N} = \left( \frac{E \cdot y}{R} \right) \times dA \quad \text{\( \because \) from (3)} \]

  \[ (4) \]

Now (5) **Moment about NN**

\[ M_{N-N} = \left( \frac{E \cdot y}{R} \right) \times dA \quad \text{\( \because \) from (3)} \]

\[ \text{for a shade Section:} \quad M = \frac{E}{R} \int (y^2 \cdot dA) \]

\[ M = \frac{E}{R} \cdot I \]

\[ \Rightarrow \quad \frac{M}{I} = \frac{E}{R} \quad \text{\( \because \) from (3)} \]

\[ \text{Equations (3) & (7)} \]

\[ \text{Finally} \quad \frac{M}{I} = \frac{E}{Y} \quad \text{\( \because \) from (3)} \]

\[ \text{Indirect Bending Eqn} \]

Mechanical Engineering REC
Moment of Resistance

• Due to the tensile and compressive stresses, forces are exerted on the layers of a beam subjected to simple bending.

• These forces will have moment about the neutral axis. The total moment of these forces about the neutral axis is known as moment of resistance of that section.

• We have seen that force on a layer of cross sectional area dA at a distance y from the neutral axis,

\[ dF = \frac{(E \times y \times dA)}{R} \]

Moment of force dF about the neutral axis= \[ dF \times y = \frac{(E \times y \times dA)}{R} \times y = \frac{E}{R} \times (y^2 dA) \].
• Hence the total moment of force about the neutral axis =

Bending moment applied = \( \int \frac{E}{R} x (y^2 \, dA) = \frac{E}{R} x I_{xx} \)

• \( I_{xx} \) is the moment of area about the neutral axis/centroidal axis.

\[
\text{Hence } M = \frac{E}{R} x I_{xx} \\
\text{Or } M/I_{xx} = \frac{E}{R} \\
\text{Hence } M/I_{xx} = \frac{E}{R} = \frac{\sigma_b}{y};
\]

\( \sigma_b \) is also known as flexural stress (\( F_b \)).

\[
\text{Hence } M/I_{xx} = \frac{E}{R} = \frac{F_b}{y}
\]

• The above equation is known as bending equation.
Condition of Simple Bending

• Bending equation is applicable to a beam subjected to pure/simple bending. Ie the bending moment acting on the beam is constant and the shear stress is zero.

• However in practical applications, the bending moment varies from section to section and the shear force is not zero.

• But in the section where bending moment is maximum, shear force (derivative of bending moment) is zero.

• Hence the bending equation is valid for the section where bending moment is maximum.
Bending of fletched beams

• A beam made up of two or more different materials assumed to be rigidly connected together and behaving like a single piece is called a flitched beam or a composite beam.

• Consider a wooden beam reinforced by steel plates. Let

\[ E_1 = \text{Modulus of elasticity of steel plate} \]
\[ E_2 = \text{Modulus of elasticity of wooden beam} \]
\[ M_1 = \text{Moment of resistance of steel plate} \]
\[ M_2 = \text{Moment of resistance of wooden beam} \]
\[ I_1 = \text{Moment of inertia of steel plate about neutral axis} \]
\[ I_2 = \text{Moment of inertia of wooden beam about neutral axis}. \]
The bending stresses can be calculated using two conditions.

- **Strain developed** on a layer at a particular distance from the neutral axis is the same for both the materials

- **Moment of resistance** of composite beam is equal to the sum of individual moment of resistance of the members
Using condition-1:

\[ \frac{\sigma_1}{E_1} = \frac{\sigma_2}{E_2}; \]
\[ \sigma_1 = \sigma_2 \times \left( \frac{E_1}{E_2} \right) \text{ or } \sigma_1 = \sigma_2 \times m; \]
where \( m = \frac{E_1}{E_2} \) is the modular ratio between steel and wood.

Using condition-2:

\[ M = M_1 + M_2; \]
\[ M_1 = \sigma_1 \times \frac{l_1}{y} \]
\[ M_1 = \sigma_2 \times \frac{l_2}{y} \]
Flitched Beams or Composite Beams

- A beam is made up of two or more different materials assumed to be rigidly connected together and behaving like a single piece is known as composite beam or wooden Flitched Beam.
Composite beams

\[
\frac{\sigma_1}{E_1} = \frac{\sigma_2}{E_2}
\]

\[
\sigma_1 = \frac{E_1}{E_2} \sigma_2 = m \sigma_2
\]

\[
M = \frac{\sigma}{y} l
\]

\[
M = M_1 + M_2 = \frac{\sigma_1}{y} l_1 + \frac{\sigma_2}{y} l_2 = \frac{\sigma_2}{y} [ml_1 + l_2]
\]

m = modular ratio
A flitched beam consists of a wooden joist 10 cm wide and 20 cm deep strengthened by two steel plates 10 mm thick and 20 cm deep as shown in Fig. 7.28. If the maximum stress in the wooden joist is 7 N/mm², find the corresponding maximum stress attained in steel. Find also the moment of resistance of the composite section. Take Young’s modulus for steel = $2 \times 10^5$ N/mm² and for wood = $1 \times 10^4$ N/mm².
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Problem 1  A steel plate of width 120 mm and of thickness 20 mm is bent into a circular arc of radius 10 m. Determine the maximum stress induced and the bending moment which will produce the maximum stress. Take $E = 2 \times 10^5$ N/mm$^2$. 
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\[ \text{Sol. Given:} \]
- Width of plate, \( b = 120 \text{ mm} \)
- Thickness of plate, \( t = 20 \text{ mm} \)

\[ \therefore \text{Moment of inertia,} \quad I = \frac{bt^3}{12} = \frac{120 \times 20^3}{12} = 8 \times 10^4 \text{ mm}^4 \]

- Radius of curvature, \( R = 10 \text{ m} = 10 \times 10^3 \text{ mm} \)
- Young’s modulus, \( E = 2 \times 10^5 \text{ N/mm}^2 \)

Let \( \sigma_{\text{max}} = \text{Maximum stress induced, and} \)
\( M = \text{Bending moment.} \)

Using equation (7.2),
\[ \frac{\sigma}{y} = \frac{E}{R} \]
\[ \therefore \sigma = \frac{E}{R} \times y \quad \ldots(i) \]
Equation (i) gives the stress at a distance \( y \) from N.A. Stress will be maximum, when \( y \) is maximum. But \( y \) will be maximum at the top layer or bottom layer.

\[
\begin{align*}
\therefore \quad y_{\text{max}} &= \frac{t}{2} = \frac{20}{2} = 10 \text{ mm.} \\
\text{Now equation (i) can be written as} \\
\sigma_{\text{max}} &= \frac{E}{R} \times y_{\text{max}} \\
&= \frac{2 \times 10^5}{10 \times 10^3} \times 10 = 200 \text{ N/mm}^2. \quad \text{Ans.}
\end{align*}
\]

From equation (7.4), we have

\[
\frac{M}{I} = \frac{E}{R}
\]

\[
\therefore \quad M = \frac{E}{R} \times I = \frac{2 \times 10^5}{10 \times 10^3} \times 8 \times 10^4
\]

\[= 16 \times 10^5 \text{ N mm} = 1.6 \text{ kNm.} \quad \text{Ans.}\]
Section Modulus for Various Shapes or Beam Sections

1. Rectangular Section

Moment of inertia of a rectangular section about an axis through its C.G. (or through N.A.) is given by,

\[ I = \frac{bd^3}{12} \]

Distance of outermost layer from N.A. is given by,

\[ y_{\text{max}} = \frac{d}{2} \]

\[ \therefore \text{ Section modulus is given by,} \]

\[ Z = \frac{I}{y_{\text{max}}} = \frac{bd^3}{12 \times \left(\frac{d}{2}\right)} = \frac{bd^3}{12} 	imes \frac{2}{d} = \frac{bd^2}{6} \]

2. Hollow Rectangular Section

Here

\[ I = \frac{BD^3}{12} - \frac{bd^2}{12} \]

\[ = \frac{1}{12} [BD^3 - bd^3] \]

\[ y_{\text{max}} = \frac{D}{2} \]

\[ \therefore \]

\[ Z = \frac{I}{y_{\text{max}}} = \frac{1}{12} [BD^3 - bd^3] \]

\[ = \frac{1}{6D} [BD^3 - bd^3] \]

\[ \cdots (7.8) \]
3. Circular Section
For a circular section,
\[ I = \frac{\pi}{64} d^4 \quad \text{and} \quad y_{\text{max}} = \frac{d}{2} \]

\[ Z = \frac{I}{y_{\text{max}}} = \frac{\pi}{64} \frac{d^4}{\left(\frac{d}{2}\right)} = \frac{\pi}{32} d^3 \]

4. Hollow Circular Section
Here
\[ I = \frac{\pi}{64} [D^4 - d^4] \]

and
\[ y_{\text{max}} = \frac{D}{2} \]

\[ Z = \frac{I}{y_{\text{max}}} = \frac{\pi}{64} \frac{[D^4 - d^4]}{\left(\frac{D}{2}\right)} \]

\[ = \frac{\pi}{32D} [D^4 - d^4] \]
Problem 2

A beam has a rectangular cross section 80 mm wide and 120 mm deep. It is subjected to a bending moment of 15 kNm at a certain point along its length. It is made from metal with a modulus of elasticity of 180 GPa. Calculate the maximum stress on the section.
Problem 3

A beam has a rectangular cross section 80 mm wide and 120 mm deep. It is subjected to a bending moment of 15 kNm at a certain point along its length. It is made from metal with a modulus of elasticity of 180 GPa. Calculate the maximum stress on the section.
Problem 4

A steel plate of width 200mm and of thickness 50mm is bent into a circular arc of radius 20m. Determine the max stress induced and the bending moment which will produce the max stress. Take $E = 2 \times 10^5$ N/mm$^2$. 
WORKED EXAMPLE No.2

A beam has a hollow circular cross section 40 mm outer diameter and 30 mm inner diameter. It is made from metal with a modulus of elasticity of 205 GPa. The maximum tensile stress in the beam must not exceed 350 MPa.

Calculate the following.

(i) the maximum allowable bending moment.
(ii) the radius of curvature.

SOLUTION

\[ D = 40 \text{ mm}, \; d = 30 \text{ mm} \]

\[ I = \frac{\pi(40^4 - 30^4)}{64} = 85.9 \times 10^3 \text{ mm}^4 = 85.9 \times 10^{-9} \text{ m}^4. \]

The maximum value of \( y \) is \( \frac{D}{2} \) so \( y = 20 \text{ mm} = 0.02 \text{ m} \)

\[ \frac{M}{I} = \frac{\sigma}{y} \]

\[ M = \frac{\sigma I}{y} = \frac{350 \times 10^6 \times 85.9 \times 10^{-9}}{0.02} = 1503 \text{ Nm} = 1.503 \text{ M Nm} \]

\[ \frac{\sigma}{y} = \frac{E}{R} \]

\[ R = \frac{Ey}{\sigma} = \frac{205 \times 10^9 \times 0.02}{350 \times 10^6} = 11.71 \text{ m} \]
Problem 4: Calculate the maximum stress induced in a cast iron pipe of external diameter 40mm, of internal diameter 20mm and of length 4m then the pipe is supported at its ends and carry a point load of 80N at its centre.

Soln:
Problem 5. A cantilever of length 2 metre fails when a load of 2 kN is applied at the free end. If the section of the beam is 40 mm $\times$ 60 mm, find the stress at the failure.
Problem 4: Calculate the maximum stress induced in a cast iron pipe of external diameter 40mm, of internal diameter 20mm and of length 4m then the pipe is supported at its ends and carry a point load of 80N at its centre.

Soln:
Problem 5. A cantilever of length 2 metre fails when a load of 2 kN is applied at the free end. If the section of the beam is 40 mm × 60 mm, find the stress at the failure.
Problem 6. A rectangular beam 200 mm deep and 300 mm wide is simply supported over a span of 8 m. What uniformly distributed load per metre the beam may carry, if the bending stress is not to exceed 120 N/mm².

**Sol.**

Given:
- Depth of beam, \( d = 200 \text{ mm} \)
- Width of beam, \( b = 300 \text{ mm} \)
- Length of beam, \( L = 8 \text{ m} \)
- Max. bending stress, \( \sigma_{\text{max}} = 120 \text{ N/mm}^2 \)

Let \( w \) = Uniformly distributed load per m length over the beam

(Fig. (a) shows the section of the beam).

Section modulus for a rectangular section is given by equation (7.7).

\[
Z = \frac{bd^2}{6} = \frac{300 \times 200^2}{6} = 2000000 \text{ mm}^3
\]

\[\text{Fig.}\]
Problem 5. A cantilever of length 2 metre fails when a load of 2 kN is applied at the free end. If the section of the beam is 40 mm × 60 mm, find the stress at the failure.

Sol. Given:
Length, \( L = 2 \text{ m} = 2 \times 10^3 \text{ mm} \)
Load, \( W = 2 \text{ kN} = 2000 \text{ N} \)
Section of beam is 40 mm × 60 mm.
\( \therefore \) Width of beam, \( b = 40 \text{ mm} \)
Depth of beam, \( d = 60 \text{ mm} \)

Section modulus of a rectangular section is given

\[ Z = \frac{bd^2}{6} = \frac{40 \times 60^2}{6} = 24000 \text{ mm}^3 \]

Maximum bending moment for a cantilever shown in Fig. 7.10 is at the fixed end.

\( \therefore \) \( M = W \times L = 2000 \times 2 \times 10^3 = 4 \times 10^6 \text{ Nmm} \)

Let \( \sigma_{\text{max}} = \text{Stress at the failure} \)
Using equation (7.6), we get
\[ M = \sigma_{\text{max}} \cdot Z \]

\( \therefore \)
\[ \sigma_{\text{max}} = \frac{M}{Z} = \frac{4 \times 10^6}{24000} = 166.67 \text{ N/mm}^2. \text{ Ans.} \]
Problem 6. A rectangular beam 200 mm deep and 300 mm wide is simply supported over a span of 8 m. What uniformly distributed load per metre the beam may carry, if the bending stress is not to exceed 120 N/mm$^2$.

Sol. Given:
Depth of beam, $d = 200$ mm
Width of beam, $b = 300$ mm
Length of beam, $L = 8$ m
Max. bending stress, $\sigma_{\text{max}} = 120$ N/mm$^2$

Let $w = \text{Uniformly distributed load per m length over the beam}$

(Fig. 6(a) shows the section of the beam).

Section modulus for a rectangular section is given by equation (7.7).

\[
Z = \frac{bd^2}{6} = \frac{300 \times 200^2}{6} = 2000000 \text{ mm}^3
\]

Max. B.M. for a simply supported beam carrying uniformly distributed load as shown in Fig. is at the centre of the beam. It is given by

\[
M = \frac{w \times L^2}{8} = \frac{w \times 8^2}{8} = 8w \text{ Nm} = 8w \times 1000 \text{ Nmm}
= 8000w \text{ Nmm}
\]

\[
(\therefore \ L = 8 \text{ m})
\]

Now using equation (7.6), we get

\[
M = \sigma_{\text{max}} \cdot Z
\]

or

\[
8000w = 120 \times 2000000
\]

\[
\therefore \ w = \frac{120 \times 2000000}{8000} = 30 \times 1000 \text{ N/m} = 30 \text{ kN/m}. \ \text{Ans.}
\]
Problem 7. A rectangular beam 300 mm deep is simply supported over a span of 4 metres. Determine the uniformly distributed load per metre which the beam may carry, if the bending stress should not exceed 120 N/mm². Take I = 8 × 10⁶ mm⁴.
Problem 7. A rectangular beam 300 mm deep is simply supported over a span of 4 metres. Determine the uniformly distributed load per metre which the beam may carry, if the bending stress should not exceed 120 N/mm². Take $I = 8 \times 10^6$ mm⁴.

**Sol.**

Given:
- Depth, $d = 300$ mm
- Span, $L = 4$ m
- Max. bending stress, $\sigma_{\text{max}} = 120$ N/mm²
- Moment of inertia, $I = 8 \times 10^6$ mm⁴

Let, $w = \text{U.D.L. per metre length over the beam}$ in N/m.

The bending stress will be maximum, where bending moment is maximum. For a simply supported beam carrying U.D.L., the bending moment is maximum at the centre of the beam [i.e., at point $C$ of Fig. 7.11 (b)]

\[
\text{Max. B.M.} = 2w \times 2 - 2w \times 1
\]

\[
= 4w - 2w
\]

\[
= 2w \text{ Nm}
\]

\[
= 2w \times 1000 \text{ Nmm}
\]

\[
M = 2000w \text{ Nmm}
\]

Now using equation (7.6), we get

\[
M = \sigma_{\text{max}} \times Z
\]

where

\[
Z = \frac{I}{y_{\text{max}}} = \frac{8 \times 10^6}{150}
\]

Hence above equation (i) becomes as

\[
2000w = 120 \times \frac{8 \times 10^6}{150}
\]

or

\[
w = \frac{120 \times 8 \times 10^6}{2000 \times 150} = 3200 \text{ N/m. Ans.}
\]
**Problem 8.** A rolled steel joist of I section has the dimensions as shown in Fig. 8.
This beam of I section carries a u.d.l. of 40 kN/m run on a span of 10 m, calculate the maximum stress produced due to bending.

**Sol.** Given:
- **u.d.l.**, \( w = 40 \text{ kN/m} = 40000 \text{ N/m} \)
- **Span**, \( L = 10 \text{ m} \)

**Moment of inertia about the neutral axis**

\[
I = \frac{200 \times 400^3}{12} - \frac{(200 - 10) \times 360^3}{12} \\
= 1066666666 - 738720000 \\
= 327946666 \text{ mm}^4
\]

**Maximum B.M. is given by,**

\[
M = \frac{w \times L^2}{8} = \frac{40000 \times 10^2}{8} \\
= 500000 \text{ Nm} \\
= 500000 \times 1000 \text{ Nmm} \\
= 5 \times 10^8 \text{ Nmm}
\]

Now using the relation,

\[
\frac{M}{I} = \frac{\sigma}{y}
\]

\[
\sigma = \frac{M}{I} \times y
\]

**\( \sigma_{max} = \frac{M}{I} \times y_{max} = \frac{5 \times 10^8}{327946666} \times 200 \) \text{ N/mm}^2. \text{ Ans.}**
Problem 8. A rolled steel joist of I section has the dimensions as shown in Fig. 
This beam of I section carries a u.d.l. of 40 kN/m run on a span of 10 m, calculate the maximum stress produced due to bending.
**Problem 8.** A rolled steel joist of I section has the dimensions: as shown in Fig. 5.

This beam of I section carries a u.d.l. of 40 kN/m run on a span of 10 m, calculate the maximum stress produced due to bending.
Problem 9. A cast iron bracket subject to bending has the cross-section of I-form with unequal flanges. The dimensions of the section are shown in Fig. 7.18. Find the position of the neutral axis and moment of inertia of the section about the neutral axis. If the maximum bending moment on the section is 40 MN mm, determine the maximum bending stress. What is the nature of the stress?
Problem 9. \ Contd.

\[
\bar{y} = \frac{6500 \times 25 + 10000 \times 150 + 10000 \times 275}{6500 + 10000 + 10000}
= \frac{162500 + 1500000 + 2750000}{26500}
= \frac{4412500}{26500} = 166.51 \text{ mm}
\]

Hence neutral axis is at a distance of 166.51 mm from the bottom face. \textbf{Ans.}
Problem 9.  

\[ \bar{y} = \frac{6500 \times 25 + 10000 \times 150 + 10000 \times 275}{6500 + 10000 + 10000} \]
\[ = \frac{162500 + 1500000 + 2750000}{26500} \]
\[ = \frac{4412500}{26500} = 166.51 \text{ mm} \]

Hence neutral axis is at a distance of 166.51 mm from the bottom face.  \textbf{Ans.}

\textbf{Moment of inertia of the section about the N.A.}

\[ I = I_1 + I_2 + I_3 \]

where \( I_1 \) = M.O.I. of bottom flange about N.A.

\[ = \text{M.O.I. of bottom flange about an axis passing through its C.G.} + A_1 \times \text{(Distance of its C.G. from N.A.)}^2 \]

\[ = \frac{130 \times 50^3}{12} + 6500 \times (166.51 - 25)^2 \]
\[ = 1354166.67 + 30163020 = 131517186.6 \text{ mm}^4 \]

\text{Similarly}

\[ I_2 = \text{M.O.I. of web about N.A.} \]
\[ = \frac{50 \times 200^3}{12} + A_2 \cdot (166.51 - y_2)^3 \]
\[ = \frac{50 \times 200^3}{12} + 10000 (166.51 - 150)^2 \]
\[ = 3333333.33 + 272580.1 \]
\[ = 33605913.43 \text{ mm}^4 \]

\text{and}

\[ I_3 = \text{M.O.I. of top flange about N.A.} \]
\[ = \frac{200 \times 50^3}{12} + A_3 \cdot (y_3 - 166.51)^3 \]
\[ = \frac{200 \times 50^3}{12} + 10000 \times (275 - 166.51)^2 \]
\[ = 2083333.33 + 117700801 = 119784134.3 \text{ mm}^4 \]
\[ \therefore I = I_1 + I_2 + I_3 = 131517186.6 + 33605913.43 + 119784134.3 \]
\[ = 284907234.9 \text{ mm}^4. \textbf{Ans.} \]

Now distance of C.G. from the upper top fibre

\[ = 300 - \bar{y} = 300 - 166.51 = 133.49 \text{ mm} \]

and the distance of C.G. from the bottom fibre

\[ = \bar{y} = 166.51 \text{ mm} \]

Hence we shall take the value of \( y = 166.51 \text{ mm} \) for maximum bending stress.

Now using the equation

\[ \frac{M}{I} \times \frac{\sigma}{y} \]

\[ \therefore \quad \sigma = \frac{M}{I} \times y = \frac{40 \times 10^6}{284907234.9} \times 166.51 = 23.377 \text{ N/mm}^2 \]

\[ \therefore \quad \text{Maximum bending stress} \]

\[ = 23.377 \text{ N/mm}^2. \quad \text{Ans.} \]

This stress will be compressive. In case of cantilevers, upper layer is subjected to tensile stress, whereas the lower layer is subjected to compressive stress.
Bending Stresses in Symmetrical Sections

- The Neutral Axis (N.A) of a symmetrical section (such as circular, rectangular or square) lies at a distance of $d/2$ from the section where $d$ is the diameter (for a circular section) or depth (for a rectangular or a square section). There is no stress at the neutral axis.
- But the stress at a point is directly proportional to its distance from the neutral axis.
- The maximum stress takes place at the outer most layer.
- For SSB there is a compressive stress above the neutral axis and a tensile stress below it.
Shear stresses in beams

- The stresses induced by shear force at a section in a beam may be analyzed as follows:

- Consider an elemental length of a beam between the sections AA and BB separated by a distance \(dx\), as shown in the following figure. Let the moments acting at AA and BB be \(M\) and \(M + dM\) respectively.

- Let CD be a fibre of thickness \(dy\) at a distance \(y\) from the neutral axis. Then bending stress at left side of the fibre \(CD = M \frac{y}{I}\)
SHEAR STRESS IN A SECTION

Area, \( A = \text{Area of } EFGH \)
Shear stresses in beams

- Force on the left side of the layer CD = $M_y (b.\,dy)/I$

- and force on the right side of the layer CD = $(M+dM)\,y\,(b.\,dy)/I$

- Therefore unbalanced force, towards right, on the layer CD is $dM\,y\,b.\,dy/I$

- There are a number of such elements above the section CD.

- Hence the unbalanced horizontal force above the section CD is

$$\int_y^t \frac{dM\,y\,b.\,dy}{y\,I}$$
This horizontal force is resisted by the resisting force provided by shearing stresses acting horizontally on the plane at CD.

Let the intensity of shear stress be \( \tau \). Equating the resisting force provided by the shearing stress to the unbalanced horizontal force we have:

\[
\tau \cdot b \cdot dx = \int_y \frac{dM}{I} \cdot y \cdot b \cdot dy
\]

or

\[
\tau = \frac{dM}{dx} \cdot \frac{1}{I} \int_y y^t \cdot y \cdot da
\]

where \( da = b \cdot dy \) is area of the element.

where the term

\[
\int_y y \cdot da = ay = \text{Moment of, area above the fibre CD about the NA.}
\]
But the term $dM / dx = F$, the shear force. Substituting in the expression for $\tau$, we obtain:

\[ \tau = \frac{Fa_y}{Ib} \]

where:

- $F =$ shear force at a section in a beam
- $a =$ area above or below a fibre (shaded area)
- $y =$ dist. from N.A. to the centroid of the shaded area
- $I =$ M.I. of the entire section about the N.A
- $b =$ breadth of the fibre.

**Note**: The above expression is for horizontal shear stress. From The principle of complementary shear, this horizontal shear stress is accompanied by a vertical shear stress of the same intensity.
1. Rectangular section
1. Rectangular section
Problem 8.3. A rectangular beam 100 mm wide and 250 mm deep is subjected to a maximum shear force of 50 kN. Determine:

(i) Average shear stress,  
(ii) Maximum shear stress, and  
(iii) Shear stress at a distance of 25 mm above the neutral axis.
• Contd...
**Problem 8.3.** A rectangular beam 100 mm wide and 250 mm deep is subjected to a maximum shear force of 50 kN. Determine:

(i) Average shear stress,

(ii) Maximum shear stress, and

(iii) Shear stress at a distance of 25 mm above the neutral axis.

**Sol.** Given:

Width, \( b = 100 \text{ mm} \)

Depth, \( d = 250 \text{ mm} \)

Maximum shear force, \( F = 50 \text{ kN} = 50,000 \text{ N} \).

(i) Average shear stress is given by,

\[
\tau_{\text{avg}} = \frac{F}{b \times d} = \frac{50,000}{100 \times 250} = 2 \text{ N/mm}^2. \quad \text{Ans.}
\]

(ii) Maximum shear stress is given by equation (8.4)

\[
\tau_{\text{max}} = 1.5 \times \tau_{\text{avg}} = 1.5 \times 2 = 3 \text{ N/mm}^2. \quad \text{Ans.}
\]

(iii) The shear stress at a distance \( y \) from N.A. is given by equation (8.2).

\[
\tau = \frac{F}{2I} \left( \frac{d^2}{4} - y^2 \right)
\]

\[
= \frac{50000}{2I} \left( \frac{250^2}{4} - 25^2 \right)
\]

\[
= \frac{50000}{2 \times \frac{b d^3}{12}} \left( \frac{62500}{4} - 625 \right) = \frac{50000 \times 12}{2 \times 100 \times 250^3} \times 15000 \text{ N/mm}^2
\]

\[
= 2.88 \text{ N/mm}^2. \quad \text{Ans.}
\]
Shear Stress Distribution in Circular Cross Section

\[ \tau = \frac{F}{EI} (R^2 - y^2) \]

At \( y = 0 \) i.e., at the neutral axis, the shear stress

\[ \tau_{\text{max}} = \frac{F}{3l} R^2 \]

\[ I = \frac{\pi}{64} \quad D^4 = \frac{\pi}{64} \times (2R)^4 \]

\[ = \frac{\pi}{4} R^4 \]

\[ \tau_{\text{max}} = \frac{F \times R^2}{3 \times \frac{\pi}{4} R^4} = \frac{4}{3} \times \frac{F}{\pi R^2} \]
SHEAR STRESS DISTRIBUTION DIAGRAM

1. Rectangular section

2. Circular section
3. **Triangular section**

4. **Hollow circular section**

\[ \tau \]

\[ \tau_{\text{max}} \]

\[ \tau_{\text{avg}} \]
5. Hollow Rectangular section

6. “I” section
7. “C” section

8. “+” section
9. “H” section

10. “T” section
Problem 8.6. A circular beam of 100 mm diameter is subjected to a shear force of 5 kN. Calculate:

(i) Average shear stress, (ii) Maximum shear stress, and (iii) Shear stress at a distance of 40 mm from N.A.

Sol. Given:

Diameter, \( D = 100 \text{ mm} \)

\[ \therefore \text{Radius, } R = \frac{100}{2} = 50 \text{ mm} \]

Shear force, \( F = 5 \text{ kN} = 5000 \text{ N} \).

(i) Average shear stress is given by,

\[ \tau_{\text{avg}} = \frac{\text{Shear force}}{\text{Area of circular section}} \]

\[ = \frac{5000}{\pi (50)^2} = 0.6366 \text{ N/mm}^2. \text{ Ans.} \]

(ii) Maximum shear stress for a circular section is given by equation (8.7).

\[ \tau_{\text{max}} = \frac{4}{3} \times \tau_{\text{avg}} \]

\[ = \frac{4}{3} \times 0.6366 = 0.8488 \text{ N/mm}^2. \text{ Ans.} \]

(iii) The shear stress at a distance 40 mm from N.A. is given by equation (8.5).

\[ \tau = \frac{F}{3I} (R^2 - y^2) \]

\[ = \frac{5000}{3 \times \frac{\pi}{64} \times 100^4} (50^2 - 40^2) \quad \left( \because y = 40 \text{ and } I = \frac{\pi}{64} \times 100^4 \right) \]

\[ = \frac{5000 \times 64}{3 \times \pi \times 100000000} (2500 - 1600) \]

\[ = 0.3055 \text{ N/mm}^2. \text{ Ans.} \]
Shear Stress Distribution of ‘I’ Section

(i) Shear stress distribution in the flange
Consider a section at a distance \( y \) from N.A. in the flange as shown in Fig. 8.8 (c).

Width of the section = \( B \)

Shaded area of flange, \( A = B \left( \frac{D}{2} - y \right) \)

Distance of the C.G. of the shaded area from neutral axis is given as

\[
\bar{y} = y + \frac{1}{2} \left( \frac{D}{2} - y \right)
\]
\[
= y + \frac{D}{4} - \frac{y}{2}
\]
\[
= \frac{D}{4} + \frac{y}{2} = \frac{1}{2} \left( \frac{D}{2} + y \right)
\]

Hence shear stress in the flange becomes,

\[
\tau = \frac{F \times A \bar{y}}{I \times B}
\]
\[
= \frac{F \times B \left( \frac{D}{2} - y \right) \times \frac{1}{2} \left( \frac{D}{2} + y \right)}{I \times B}
\]
\[
= \frac{F}{2I} \left[ \left( \frac{D}{2} \right)^2 - y^2 \right]
\]
\[
= \frac{F}{2I} \left( \frac{D^2}{2} - y^2 \right)
\]

Hence, the variation of shear stress (\( \tau \)) with respect to \( y \) in the flange is parabolic. It is also clear from equation (8.8) that with the increase of \( y \), shear stress decreases.

(a) For the upper edge of the flange,

\[
y = \frac{D}{2}
\]

Hence shear stress,

\[
\tau = \frac{F}{2I} \left[ \frac{D^2}{4} - \left( \frac{D}{2} \right)^2 \right] = 0.
\]
(b) For the lower edge of the flange,
\[ y = \frac{d}{2} \]

Hence
\[ \tau = \frac{F}{2I} \left[ \frac{D^3}{4} - \left( \frac{d}{2} \right)^2 \right] = \frac{F}{2I} \left( \frac{D^2}{4} - \frac{d^2}{4} \right) \]
\[ = \frac{F}{8I} (D^2 - d^2) \]  \hspace{1cm} (8.9)

(ii) Shear stress distribution in the web

Consider a section at a distance \( y \) in the web from the N.A. as shown in Fig. 8.9.

Width of the section = \( b \).

Here \( A\bar{y} \) is made up of two parts i.e., moment of the flange area about N.A. plus moment of the shaded area of the web about the N.A.

\[ A\bar{y} = \text{Moment of the flange area about N.A.} + \text{moment of the shaded area of web about N.A.} \]
\[ = B \left( \frac{D}{2} - \frac{d}{2} \right) \times \frac{1}{2} \left( \frac{D}{2} + \frac{d}{2} \right) + b \left( \frac{d}{2} - y \right) \times \frac{1}{2} \left( \frac{d}{2} + y \right) \]
\[ = \frac{B}{8} (D^2 - d^2) + \frac{b}{2} \left( \frac{d^2}{4} - y^2 \right) \]

Hence the shear stress in the web becomes as
\[ \tau = \frac{F \times A\bar{y}}{I \times b} = \frac{F}{I \times b} \times \left[ \frac{B}{8} (D^2 - d^2) + \frac{b}{2} \left( \frac{d^2}{4} - y^2 \right) \right] \]  \hspace{1cm} (8.10)

From equation (8.10), it is clear that variation of \( \tau \) with respect to \( y \) is parabolic. Also with the increase of \( y \), \( \tau \) decreases.

At the neutral axis, \( y = 0 \) and hence shear stress is maximum.

\[ \tau_{\text{max}} = \frac{F}{I \times b} \left[ \frac{B}{8} (D^2 - d^2) + \frac{b}{2} \left( \frac{d^2}{4} \right) \right] \]
\[ = \frac{F}{I \times b} \left[ \frac{B}{8} (D^2 - d^2) + \frac{bd^2}{8} \right] \]  \hspace{1cm} (8.11)

At the junction of top of the web and bottom of flange,
\[ y = \frac{d}{2} \]

Hence shear stress is given by,
\[ \tau = \frac{F \times B \times (D^2 - d^2)}{8I \times b} \]  \hspace{1cm} (8.12)
Problem 8.7. An I-section beam 350 mm × 150 mm has a web thickness of 10 mm and a flange thickness of 20 mm. If the shear force acting on the section is 40 kN, find the maximum shear stress developed in the I-section.
**Problem 8.7.** An I-section beam 350 mm × 150 mm has a web thickness of 10 mm and a flange thickness of 20 mm. If the shear force acting on the section is 40 kN, find the maximum shear stress developed in the I-section.

**Sol. Given:**

Overall depth, \( D = 350 \text{ mm} \)

Overall width, \( B = 150 \text{ mm} \)

Web thickness, \( b = 10 \text{ mm} \)

Flange thickness, \( d = 20 \text{ mm} \)

.: Depth of web, \( d = 350 - (2 \times 20) = 310 \text{ mm} \)

Shear force on the section, \( F = 40 \text{ kN} = 40,000 \text{ N} \)

Moment of inertia of the section about neutral axis,

\[
I = \frac{150 \times 350^3}{12} - \frac{140 \times 310^3}{12} \text{ mm}^4
\]

\[
= 535937500 - 347561666.6
\]

\[
= 188375833.4 \text{ mm}^4.
\]

Maximum shear stress is given by equation (8.11)

\[
\tau_{\text{max}} = \frac{F}{I \times b} \left[ \frac{B(D^2 - d^2)}{8} + \frac{bd^2}{8} \right]
\]

\[
= \frac{40000}{188375833.4 \times 10} \left[ \frac{150(350^2 - 310^2)}{8} + \frac{10 \times 310^2}{8} \right]
\]

\[
= 0.000021234 \left[ \frac{150}{8} (122500 - 96100) + 120125 \right]
\]

\[
= 13.06 \text{ N/mm}^2. \text{ Ans.}
\]
**Problem 8.8.** For the problem 8.7, sketch the shear stress distribution across the section. Also calculate the total shear force carried by the web.

**Sol.** Given:
From problem 8.7, we have

\[
B = 150 \text{ mm} ; \quad D = 350 \text{ mm} \\
d = 310 \text{ mm} ; \quad b = 10 \text{ mm} \\
F = 40000 \text{ N} ; \quad I = 188.375 \times 10^6 \text{ mm}^4 \\
\tau_{\text{max}} = 13.06 \text{ N/mm}^2.
\]

*Shear stress distribution in the flange*

The shear stress at the upper edge of the flange is zero.

Actually shear stress distribution in the flange is given by equation (8.8) as

\[
\tau = \frac{F}{2I} \left( \frac{D^2}{4} - y^2 \right)
\]
**Problem 8.8.** For the problem 8.7, sketch the shear stress distribution across the section. Also calculate the total shear force carried by the web.

**Sol.** Given:
From problem 8.7, we have

\[ B = 150 \text{ mm} \; ; \; D = 350 \text{ mm} \]
\[ d = 310 \text{ mm} \; ; \; b = 10 \text{ mm} \]
\[ F = 40000 \text{ N} \; ; \; I = 188.375 \times 10^6 \text{ mm}^4 \]
\[ \tau_{\text{max}} = 13.06 \text{ N/mm}^2. \]

**Shear stress distribution in the flange**

The shear stress at the upper edge of the flange is zero.

Actually shear stress distribution in the flange is given by equation (8.8) as

\[ \tau = \frac{F}{2I} \left( \frac{D^2}{4} - y^2 \right) \]
**Problem 8.8.** For the problem 8.7, sketch the shear stress distribution across the section. Also calculate the total shear force carried by the web.

**Sol.** Given:

From problem 8.7, we have

\[
B = 150 \text{ mm} ; \quad D = 350 \text{ mm} \\
d = 310 \text{ mm} ; \quad b = 10 \text{ mm} \\
F = 40000 \text{ N} ; \quad I = 188.375 \times 10^6 \text{ mm}^4 \\
\tau_{\text{max}} = 13.06 \text{ N/mm}^2.
\]

*Shear stress distribution in the flange*

The shear stress at the upper edge of the flange is zero.

Actually, shear stress distribution in the flange is given by equation (8.8) as

\[
\tau = \frac{F}{2I} \left( \frac{D^2}{4} - y^2 \right)
\]

\[\text{...}(i)\]
For the upper edge of the flange,

\[ y = \frac{D}{2} \]

\[ \tau = \frac{F}{2I} \left( \frac{D^2}{4} - \left( \frac{D}{2} \right)^2 \right) = \frac{F}{2I} \left( \frac{D^2}{4} - \frac{D^2}{4} \right) = 0 \]

For the lower edge of the upper flange (i.e.,) at the joint of web and flange,

\[ y = \frac{d}{2} \]

\[ \therefore \text{ Substituting this value in equation (i), we get} \]

\[ \tau = \frac{F}{2I} \left( \frac{D^2}{4} - \left( \frac{d}{2} \right)^2 \right) = \frac{F}{2I} \left( \frac{D^2}{4} - \frac{d^2}{4} \right) \]

\[ = \frac{F}{8I} (D^2 - d^2) = \frac{40000}{8 \times 188.375 \times 10^6} (350^2 - 310^2) \]

\[ = 0.7007 \text{ N/mm}^2. \]

**Shear stress distribution in the web**

The shear stress is maximum at N.A. and it is given by,

\[ \tau_{\text{max}} = 13.06 \text{ N/mm}^2 \]

 calculat in problem 8.7)

The shear stress at the junction of web and flange is given by equation (8.12) as

\[ \tau = \frac{F \times B}{8I \times b} (D^2 - d^2) \]

\[ = \frac{40000 \times 150}{8 \times 188.375 \times 10^6 \times 10} (350^2 - 310^2) = 10.51 \text{ N/mm}^2 \]

(The shear stress at the junction can also be obtained as equal to

\[ \frac{B}{b} \times 0.7007 = \frac{150}{10} \times 0.7007 = 10.51 \text{ N/mm}^2 \]
**T-Section.** The shear stress distribution over a T-section is obtained in the same manner as over an I-section. But in this case the position of neutral axis (i.e., position of C.G.) is to be obtained first, as the section is not symmetrical about x-x axis. The shear stress distribution diagram will also not be symmetrical.

**Problem**  
The shear force acting on a section of a beam is 50 kN. The section of the beam is of T-shaped of dimensions 100 mm × 100 mm × 20 mm as shown in Fig. 8.12. The moment of inertia about the horizontal neutral axis is $314.221 \times 10^4$ mm$^4$. Calculate the shear stress at the neutral axis and at the junction of the web and the flange.
A beam of I-section is having overall depth as 500 mm and overall width as 190 mm. The thickness of flanges is 25 mm whereas the thickness of the web is 15 mm. The moment of inertia about N.A. is given as $6.45 \times 10^8$ mm$^4$. If the section carries a shear force of 40 kN, calculate the maximum shear stress. Also sketch the shear stress distribution across the section.